Higher Mathematics

Wave Function - Solutions

Marks are indicated in brackets after each question number

2013 Paper 1 Question 23, (4) (2)

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x-a) = k\sin x\cos a - k\cos x\sin a$

$$= k\cos a\sin x - k\sin a\cos x$$

By inspection we have $\sqrt{3} = k\cos a$ and $1 = k\sin a$

$$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\frac{k\sin a}{k\cos a} = \frac{1}{\sqrt{3}} = \tan a$$

Giving $a = 30°$ by exact values.

Since we want a to be such that both $\sin a$ and $\cos a$ are positive, by CAST, a must be in quadrant 1. So, $30°$ is correct.

So, $\sqrt{3}\sin x - \cos x = 2\sin(x-30)\,°$

b) $4 + 5\cos x - 5\sqrt{3}\sin x = 4 - 5(\sqrt{3}\sin x - \cos x)$

$= 4 - 5 \cdot 2\sin(x-30)$ from a)

$$= 4 - 10\sin(x-30)$$

$$= -10\sin(x-30) + 4$$

$-10\sin(x-30)$ has a maximum value of 10.

So, $-10\sin(x-30) + 4$ has a maximum value of 14.

2014 Paper 1 Question 4, (2)

Let $3\sin x - 4\cos x = k\cos(x-a)$

$$= k\cos x\cos a + k\sin x\sin a$$

$$= k\cos a\cos x + k\sin a\sin x$$

By inspection $3 = k\sin a$ and $-4 = k\cos a$

2015 Paper 2 Question 9, (8)

$$\text{Let } 36\sin(1.5t) - 15\cos(1.5t) = k\sin(1.5t - a)$$
$$= k\sin(1.5t)\cos a - k\cos(1.5t)\sin a$$
$$= k\cos a\sin(1.5t) - k\sin a\cos(1.5t)$$

By inspection we have $36 = k\cos a$ and $15 = k\sin a$

$k = \sqrt{36^2 + 15^2} = 39$

$$\frac{k\sin a}{k\cos a} = \frac{15}{36} = \tan a$$

$a = tan^{-1}\left(\frac{15}{36}\right) = 0.39$ radians (confirm the correct quadrant using CAST)

So, $36\sin(1.5t) - 15\cos(1.5t) = 39\sin(1.5t - 0.39)$

$h = 36\sin(1.5t) - 15\cos(1.5t) + 65$

$\quad = 39\sin(1.5t - 0.39) + 65$ (from above)

Let $h = 100$ to give

$100 = 39\sin(1.5t - 0.39) + 65$

$35 = 39\sin(1.5t - 0.39)$

$$\frac{35}{39} = \sin(1.5t - 0.39)$$

$1.5t - 0.39 = \sin^{-1}\left(\frac{35}{39}\right) = 1.11$ radians

$1.5t = 1.5, t = 1$

Second solution for $\frac{35}{39} = \sin(1.5t - 0.39)$ is given by

$\pi - 1.11 = 2.03$ radians, so

$1.5t - 0.39 = 2.03$

$t = 1.61$

So, the solutions are $t = 1$ second and $t = 1.61$ seconds.

2016 Paper 2 Question 8, (4) (4)

a) Let $5\cos x - 2\sin x = k\cos(x + a)$

$= k\cos x\cos a - k\sin a\sin x$

$= k\cos a\cos x - k\sin a\sin x$

By inspection $k\cos a = 5$ and $k\sin a = 2$

$k = \sqrt{5^2 + 2^2} = \sqrt{29}$

$\frac{k\sin a}{k\cos a} = \frac{2}{5} = \tan a$

$a = tan^{-1}\left(\frac{2}{5}\right) = 0.38$ radians

So, $5\cos x - 2\sin x = \sqrt{29}\cos(x + 0.38)$

b) Equating the line and curve gives

$10 + 5\cos x - 2\sin x = 12$

$5\cos x - 2\sin x = 2$

Using the result from part a) we have

$\sqrt{29}\cos(x + 0.38) = 2$

$\cos(x + 0.38) = \frac{2}{\sqrt{29}}$

First solution

$x + 0.38 = cos^{-1}(\frac{2}{\sqrt{29}})$

$x + 0.38 = 1.19$

$x = 0.81$ radians

Second solution

$x + 0.38 = 2\pi - 0.81$

$x = 4.71$ radians

2017 Paper 1 Question 14, (4) (3)

a) Let $\sqrt{3}\sin x - \cos x = k\sin(x - a)$

$$= k\sin x\cos a - k\cos x\sin a$$

$$= k\cos a\sin x - k\sin a\cos x$$

By inspection $\sqrt{3} = k\cos a$ and $1 = k\sin a$

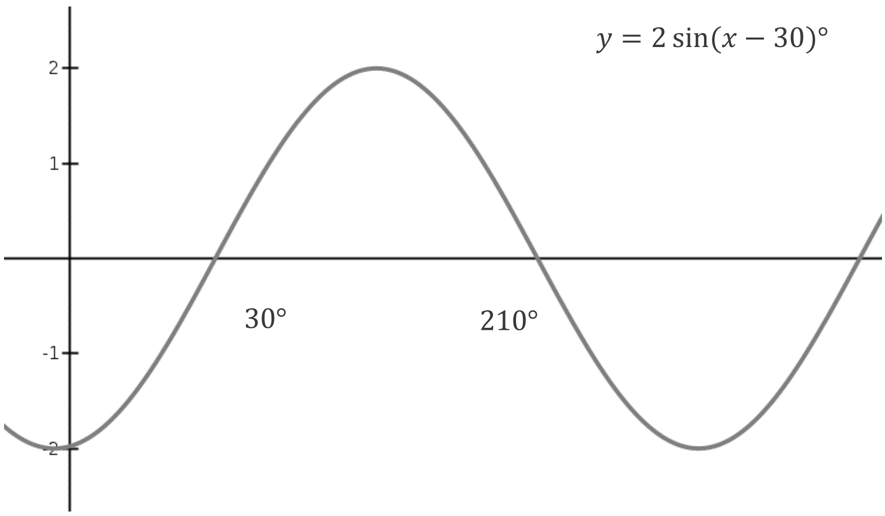
$$k = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\frac{k\sin a}{k\cos a} = \frac{1}{\sqrt{3}} = \tan a$$

$a = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$ (Confirm this is in the correct quadrant using CAST)

So, $\sqrt{3}\sin x - \cos x = 2\sin(x - 30)°$

b) Since $\sqrt{3}\sin x - \cos x = 2\sin(x - 30)°$ the graph of $y = \sqrt{3}\sin x - \cos x$ is the same as the graph of $y = 2\sin(x - 30)$.



2018 Paper 2 Question 8, (4) (1) (2)

a) $2\cos x - \sin x = k\cos(x - a)$

$$= k\ cos\ x \cos a + k \sin x \sin a$$

$$= k\cos a \cos x + k \sin a \sin x$$

$2 = k\cos a$

$-1 = k\sin a$

$k = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$\tan a = -\frac{1}{2}$

$tan^{-1}\left(\frac{1}{2}\right) = 26.4°$

From CAST $a = 360 - 26.4 = 333.6°$

$2\cos x - \sin x = \sqrt{5}\cos(x - 333.6)°$

b) i) $6\cos x - 3\sin x = 3(2\cos x - \sin x)$

$$= 3\sqrt{5}\cos(x - 333.6)°$$

Considering the graph of this function gives a minimum value of $-3\sqrt{5}$.

ii) $3\sqrt{5}\cos(x - 333.6)° = -3\sqrt{5}$

$\cos(x - 333.6)° = -1$

$x - 333.6 = 180$

$x = 180 + 333.6 = 513.4$

$x = 513.4 - 360 = 153.4°$

2019 Paper 2 Question 6, (4) (3)

a) $2\cos x - 3\sin x = k\cos\ (x + a)$

$$= k\cos x\cos a - k\sin x\sin a$$

$$= k\cos a\cos x - k\sin a\sin x$$

$2 = k\cos a \qquad\qquad 3 = k\sin a$

$k = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\frac{3}{2} = \tan a$

$a = tan^{-1}\left(\frac{3}{2}\right) = 56.3°$ (Check that this is in the correct quadrant using CAST)

So, $2\cos x - 3\sin x = \sqrt{13}\cos\ (x + 56.3)°$

b) $2\cos x - 3\sin x = 3$

Using the result from a) gives

$\sqrt{13}\cos\ (x + 56.3)° = 3$

$\cos(x + 56.3) = \frac{3}{\sqrt{13}}$

$cos^{-1}\left(\frac{3}{\sqrt{13}}\right) = 33.69°$

Using CAST, solutions to $\cos(x + 56.3) = \frac{3}{\sqrt{13}}$ should lie in quadrants 1 & 4

$x + 56.3 = 33.69$

$x = 33.69 - 56.3 = -22.61$ which is not in the range $0 \le x \le 360°$

But since Cosine repeats values every 360° we have a solution given by

$x = -22.61 + 360 = 337.39°$

$x + 56.3 = 360 - 33.69$

$x = 270°$

2022 Paper 2 Question 3, (4) (3)

a) $4\sin x + 5\cos x = k\sin(x+a)$

$$= k\sin x\cos a + k\cos x\sin a$$

$$= k\ cos\ a\sin x + k\sin a\cos x$$

Equating gives

$k\cos a = 4$ and $k\sin a = 5$

$k = \sqrt{4^2 + 5^2} = \sqrt{41}$

$\tan a = \frac{5}{4}$

$a = tan^{-1}\left(\frac{5}{4}\right) = 0.896$

So, $4\sin x + 5\cos x = \sqrt{41}\sin(x + 0.896)$

b) $4\sin x + 5\cos x = 5.5$

Using the result from part **a)** gives

$\sqrt{41}\sin(x + 0.896) = 5.5$

$\sin(x + 0.896) = \frac{5.5}{\sqrt{41}}$

$sin^{-1}\left(\frac{5.5}{\sqrt{41}}\right) = 1.033$

Using CAST we have $\pi - 1.033 = 2.108$

So, $x + 0.896 = 1.033$ and $x + 0.896 = 2.108$

Giving $x = 0.137$ and $x = 1.212$

2023 Paper 2 Question 9, (4) (1) (2)

a) $7\cos x - 3\sin x = k\sin(x + a)$

$$= k\sin x\cos a + k\cos x\sin a$$

$$= k\cos a\sin x + k\sin a\cos x$$

$\cos a = -3$

$\sin a = 7$

$k = \sqrt{(-3)^2 + 7^2} = \sqrt{58}$

$\tan a = \frac{7}{-3}$

$tan^{-1}\left(\frac{7}{3}\right) = 66.8°$

Using CAST $a = 180 - 66.8 = 113.2°$

$7\cos x - 3\sin x = \sqrt{58}\sin(x + 113.2)°$

b) i) $14\cos x - 6\sin x = 2(7\cos x - 3\sin x)$

$$= 2\sqrt{58}\sin(x + 113.2)°$$

The maximum value of this function is $2\sqrt{58}$.

ii) The maximum value for the Sine function occurs at $x = 90°$.

Since $x + 113.2$ is a translation of the graph to the left by $113.2°$ the point is $90 - 113.2 = -23.2$. But since this is outside the range the answer is given by $-23.2 + 360 = 336.8°$.

2024 Paper 1 Question 11, (4) (3)

a) $\cos x + \sqrt{3}\sin x = k\cos(x - a)$

$$= k\cos x\cos a + k\sin x\sin a$$

$$= k\cos a\cos x + k\sin a\sin x$$

$k\cos a = 1$ and $k\sin a = \sqrt{3}$

$k = \sqrt{1^2 + (\sqrt{3})^2} = 2$

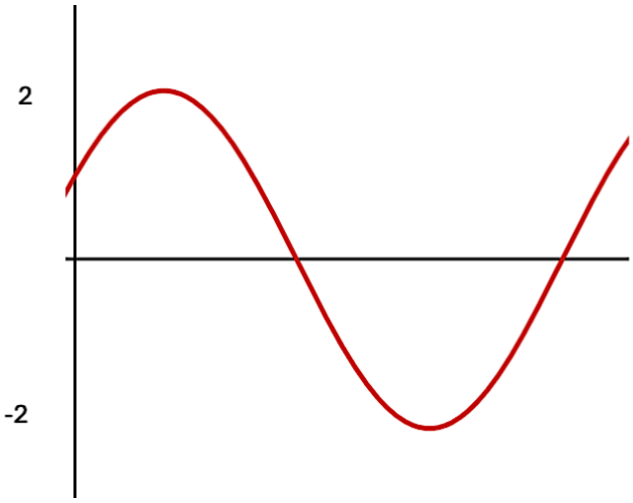
$\tan a = \frac{\sqrt{3}}{1} = \sqrt{3}$

Using CAST $a = 60°$

So, $\cos x + \sqrt{3}\sin x = 2\cos(x - 60)°$

b) $\cos x + \sqrt{3}\sin x = 2\cos(x - 60)°$

So, we have to sketch the graph of $y = 2\cos(x - 60)°$



2025 Paper 2 Question 6, (4) (3)

a) $5\cos x - 9\sin x = k\cos(x + a)$

$$= k\cos x\cos a - k\sin x\sin a$$

$$= k\cos a\cos x - k\sin a\sin x$$

$k\cos a = 5$ and $k\sin a = 9$

$k = \sqrt{5^2 + 9^2} = \sqrt{106}$

$\tan a = \frac{9}{5}$ giving $a = tan^{-1}\left(\frac{9}{5}\right) = 1.06$

So, $5\cos x - 9\sin x = \sqrt{106}\cos(x + 1.06)$

b) $5\cos x - 9\sin x = 7$

$\sqrt{106}\cos(x + 1.06) = 7$

$\cos(x + 1.06) = \frac{7}{\sqrt{106}}$

$cos^{-1}\left(\frac{7}{\sqrt{106}}\right) = 0.82$

So, $x + 1.06 = 0.82$ giving $x = 0.82 - 1.06 = -0.24$

-0.24 is outside of the range of values. However, we can add 2π to give

$x = -0.24 + 2\pi = 6.04$

Using CAST we have $2\pi - 0.82 = 5.46$

$x + 1.06 = 5.46$

$x = 5.46 - 1.06 = 4.4$

2026 Paper 2 Question 13, (4) (3)

a) $2\sin x - 7\cos x = k\sin(x - a)$

$$= k\sin x\cos a - k\cos x\sin a$$

$$= k\cos a\sin x - k\sin a\cos x$$

$k\cos a = 2$ and $-k\sin a = -7$ giving $k\sin a = 7$

$k = \sqrt{2^2 + 7^2} = \sqrt{53}$

$\tan a = \frac{7}{2}$

$tan^{-1}\left(\frac{7}{2}\right) = 74°$

Using CAST $a = 74°$

So, $2\sin x - 7\cos x = \sqrt{53}\sin(x - 74)°$

b) Equate the two functions to give

$2\sin x - 7\cos x = 3$

$\sqrt{53}\sin(x - 74) = 3$

$\sin(x - 74) = \frac{3}{\sqrt{53}}$

$sin^{-1}\left(\frac{3}{\sqrt{53}}\right) = 24.34°$

From CAST we also have a solution given by $180 - 24.34 = 155.66°$

$x - 74 = 24.34$ giving $\quad x = 24.34 + 74 = 98.34°$

$x - 74 = 155.66$ giving $\quad x = 155.66 + 74 = 229.66°$

From the diagram it is clear that neither of these are the x co-ordinate of P so we

Have to subtract the period of the Sine function, 180°, to give:

$x = 229.66 - 360 = -130.34°$

So, $P = (-130.34, 3)$