Higher Mathematics

Recurrence Relations - Solutions

Marks are indicated in brackets after each question number

2013 Paper 1 Question 8, (2)

$u_{n+1} = 0.1u_n + 8$

$u_1 = 0.1u_0 + 8$

$11 = 0.1u_0 + 8$

$3 = 0.1u_0$

$30 = u_0$

Since $-1 < 0.1 < 1$ the sequence does have a limit as n$\rightarrow \infty$.

The answer is C) Only statement 2) is correct.

2013 Paper 2 Question 1, (4)

$u_1 = 4, u_2 = 7, u_3 = 16$

$u_{n+1} = mu_n + c$

Substituting u_1 and u_2 gives

$u_2 = mu_1 + c$

$7 = 4m + c$

Substituting u_2 and u_3 gives

$u_3 = mu_2 + c$

$16 = 7m + c$

Now we have simultaneous equations to solve for m & c

$7 = 4m + c \qquad (1)$

$16 = 7m + c \qquad (2)$

(2) – (1) gives

$9 = 3m$

$m = 3$

Substituting $m = 3$ into (2) gives

$16 = 7 \cdot 3 + c$

$c = -5$

So, $m = 3, c = -5$.

2014 Paper 1 Question 1, (2)

$$u_{n+1} = \frac{1}{3}u_n + 1$$

$$u_3 = \frac{1}{3}u_2 + 1 = \frac{1}{3}\cdot 15 + 1 = 6$$

$$u_4 = \frac{1}{3}u_3 + 1 = \frac{1}{3}\cdot 6 + 1 = 3$$

2014 Paper 1 Question 10, (2)

For $u_{n+1} = au_n + b$ the limit occurs where $-1 < a < 1$

So, limit occurs where $-1 < (k-2) < 1$

Adding 2 to both sides gives $1 < k < 3$

2015 Paper 2 Question 3, (1) (5)

a) $t_2 = \frac{3}{4}t_1 + 13$

$= \frac{3}{4}\cdot 13 + 13$

$= \frac{39}{4} + \frac{52}{4}$

$= \frac{91}{4}$

b) $f_{n+1} = \frac{1}{3}f_n + 32$

Limit $= \frac{32}{1-\frac{1}{3}} = \frac{32}{\frac{2}{3}} = 48$

Since $48 < 50$ the frog does not escape from the well.

$t_{n+1} = \frac{3}{4}t_n + 13$

Limit $= \frac{13}{1-\frac{3}{4}} = \frac{13}{\frac{1}{4}} = 52$

Since $52 > 50$ the toad does escape from the well.

2016 Paper 1 Question 3, (1) (1) (2)

a) $u_{n+1} = \frac{1}{3}u_n + 10$

$u_4 = \frac{1}{3}u_3 + 10$

$u_4 = \frac{1}{3} \cdot 6 + 10$

$u_4 = 12$

b) Since $-1 < \frac{1}{3} < 1$ the sequence has a limit as $n \to \infty$.

c) Limit $= \frac{10}{1-\frac{1}{3}} = \frac{10}{\frac{2}{3}} = 15$

2017 Paper 1 Question 9, (2) (1)

a) $u_{n+1} = mu_n + 6$

$u_2 = mu_1 + 6$

$13 = 28m + 6$

$7 = 28m$

$m = \frac{1}{4}$

b) i) Since $-1 < \frac{1}{4} < 1$ the sequence has a limit as $n \to \infty$.

ii) Limit $= \frac{6}{1-\frac{1}{4}} = \frac{6}{\frac{3}{4}} = 8$

2017 Paper 2 Question 8, (2) (4)

a) $u_{n+1} = ku_n - 20$

$u_1 = k \cdot u_0 - 20 = 5k - 20$

$u_2 = k \cdot u_1 - 20 = k(5k - 20) - 20$

$\quad = 5k^2 - 20k - 20$

b) Let $u_2 < u_0$ to give

$5k^2 - 20k - 20 < 5$

$5k^2 - 20k - 25 < 0$

Let $5k^2 - 20k - 25 = 0$

$$k^2 - 4k - 5 = 0$$
$$(k-5)(k+1) = 0$$
$$k = -1, k = 5$$

So, $5k^2 - 20k - 25 < 0$ when $-1 \leq k \leq 5$.

2018 Paper 2 Question 7, (2) (2) (1) (3) (1)

a) i)

2	2	-3	-3	2
		4	2	-2
	2	1	-1	0

Zero remainder shows that $(x-2)$ is a factor.

ii) $2x^3 - 3x^2 - 3x + 2 = (x-2)(2x^2 + x - 1)$
$= (x-2)(2x-1)(x+1)$

b) $u_5 = 2a - 3$

$u_6 = au_5 - 1$
$= a(2a-3) - 1$
$= 2a^2 - 3a - 1$

$u_7 = au_6 - 1$
$= a(2a^2 - 3a - 1) - 1$
$= 2a^3 - 3a^2 - a - 1$

c) i) $2a - 3 = 2a^3 - 3a^2 - a - 1$

$2a^3 - 3a^2 - 3a + 2 = 0$

From a) ii) we have

$(a-2)(2a-1)(a+1) = 0$

$a = -1, \frac{1}{2}, 2$

But since there is a limit, $-1 < a < 1$

So, $a = \frac{1}{2}$.

ii) Limit $= \frac{b}{1-a} = \frac{-1}{1-\frac{1}{2}} = -2$

2019 Paper 1 Question 4, (3) (1)

a) Let $u_0 = 6, u_1 = 9, u_2 = 11$

$u_1 = mu_0 + c$

$9 = 6m + c$

$c = 9 - 6m \qquad (1)$

$u_2 = mu_1 + c$

$11 = 9m + c \qquad (2)$

Substitute (1) into (2) giving

$11 = 9m + 9 - 6m$

$2 = 3m$

$m = \frac{2}{3}$

From 1) $c = 9 - 6\left(\frac{2}{3}\right) = 5$

b) $u_3 = \frac{2}{3}(11) + 5$

$= \frac{22}{3} + \frac{15}{3}$

$= \frac{37}{3}$

$u_4 = \frac{2}{3}\left(\frac{37}{3}\right) + 5$

$= \frac{74}{9} + 5$

$= \frac{74}{9} + \frac{45}{9}$

$= \frac{119}{9}$

2019 Paper 2 Question 4, (1) (1) (2)

a) $U_{n+1} = aU_n + b$

$U_{n+1} = 0.973U_n + 30$

$a = 0.973 \qquad b = 30$

b) i) Since $-1 < 0.973 < 1$ the recurrence relation generates a sequence which has a limit, so the mouse population will tend towards that limit over time.

ii) Limit $= \frac{30}{1-0.973} = 1{,}111.111$

$= 1{,}100$ to the nearest hundred.

So, the long term population will be 1,100 to the nearest hundred.

2024 Paper 1 Question 2, (1) (1) (2)

a) $u_2 = \frac{1}{5}u_1 + 12$

$= \frac{1}{5}(20) + 12$

$= 16$

b) i) Since $-1 < \frac{1}{5} < 1$

ii) Limit $= \frac{12}{1-\frac{1}{5}}$

$= \frac{12}{\frac{4}{5}}$

$= 15$

2025 Paper 2 Question 9, (2) (1)

a) $a = m$ and $b = 4$

Limit $= \frac{b}{1-a} = 10$

$\frac{4}{1-m} = 10$

$4 = 10(1-m)$

$4 = 10 - 10m$

$10m = 6$

$m = \frac{6}{10} = \frac{3}{5}$

b) $u_{n+1} = \frac{3}{5}u_n + 4$

$u_1 = \frac{3}{5}u_0 + 4$

$19 = \frac{3}{5}u_0 + 4$

$15 = \frac{3}{5}u_0$

$u_0 = 25$

2026 Paper 2 Question 5, (1) (1) (2)

a) $u_1 = 1.125\, u_0 + 5$

$u_1 = (1.125 \times 2) + 5$

$u_1 = 7.25$

b) Since 1.125 does not lie in the range -1 to 1 the sequence does not approach a limit.

c) $u_2 = 1.125\, u_1 + 5 = 7.25$

$u_3 = 1.125\, u_2 + 5 = 13.16$

$u_4 = 1.125\, u_3 + 5 = 19.8$

$u_5 = 1.125\, u_4 + 5 = 27.26$

$u_6 = 1.125\, u_5 + 5 = 35.67$

So, $n = 6$ gives $u_n > 30$