Higher Mathematics

Optimization - Solutions

Marks are indicated in brackets after each question number

2013 Paper 2 Question 7, (3) (7)

a) $L = 3x + 4y$ by inspection of the diagram

Area $= 24 = 2xy$

$$y = \frac{24}{2x} = \frac{12}{x}$$

Substituting $y = \frac{12}{x}$ into L gives

$$L = 3x + 4 \cdot \frac{12}{x} = 3x + \frac{48}{x}$$

b) i) $L(x) = 3x + \frac{48}{x}$

$$= 3x + 48x^{-1}$$

$$L'(x) = 3 - 48x^{-2}$$

$$= 3 - \frac{48}{x^2}$$

Minimum values occur where $L'(x) = 0$

$$3 - \frac{48}{x^2} = 0$$

$$3x^2 - 48 = 0$$

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

$$x = -4, x = 4$$

Since x is a length it cannot be negative, so $x = 4$.

x	1	4	10
$L'(x)$	$+$	0	$-$
Shape	\	—	/

$x = 4$ gives a minimum value of $L(x)$.

ii) Minimum length occurs where $x = 4$.

$L(4) = 3 \cdot 4 + \frac{48}{4} = 24m$

Minimum cost $= 24 \cdot £8.25 = £198$.

2015 Paper 2 Question 8, (1) (1) (8)

a) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

i) If the crocodile does not travel on land then $x = 20$.

So, $T(20) = 5\sqrt{36 + 20^2} + 4(20 - 20)$

$= 5\sqrt{436} \cong 104$ tenths of a second, 10.4 seconds.

ii) If the crocodile swims the shortest distance then $x = 0$

So, $T(0) = 5\sqrt{36 + 0^2} + 4(20 - 0)$

$= 110$ tenths of a second, 11 seconds

b) $T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

$= 5(36 + x^2)^{\frac{1}{2}} + 80 - 4x$

$T'(x) = [\frac{5}{2}(36 + x^2)^{-\frac{1}{2}} \cdot 2x] - 4$

$= 5x(36 + x^2)^{-\frac{1}{2}} - 4$

$= \frac{5x}{(36 + x^2)^{\frac{1}{2}}} - 4$

Minimum values occur where $T'(x) = 0$.

Let $\frac{5x}{(36+x^2)^{\frac{1}{2}}} - 4 = 0$

$$\frac{5x}{(36+x^2)^{\frac{1}{2}}} = 4$$

$$4(36 + x^2)^{\frac{1}{2}} = 5x$$

$$(36 + x^2)^{\frac{1}{2}} = \frac{5x}{4}$$

Squaring both sides gives

$$36 + x^2 = \frac{25x^2}{16}$$

$$576 + 16x^2 = 25x^2$$

$$9x^2 - 576 = 0$$

$$x^2 - 64 = 0$$

$$x = 8, x = -8$$

Since x is a distance it cannot be negative so $x = 8$.

We must prove that $= 8$ gives a minimum value of $T(x)$ using a nature table.

x	1	8	10
$T'(x)$	$-$	0	$+$
Shape	$\searrow$	$\rightarrow$	$\nearrow$

So $x = 8$ does give a minimum value of $T(x)$

$T(8) = 5\sqrt{36+8^2} + 4(20-8)$

$= 98$

So, the minimum time possible is 98 tenths of a second, 9.8 seconds.

2016 Paper 2 Question 7, (3) (6)

a) Length $= 9x + 8y$

Area $= 2y \cdot 3x = 108$

$$6xy = 108$$

$$y = \frac{18}{x}$$

Substituting $y = \frac{18}{x}$ into Length gives

Length $= 9x + 8 \cdot \frac{18}{x}$

$= 9x + \frac{144}{x}$

So, $L(x) = 9x + \frac{144}{x}$

b) Minimum values of $L(x)$ occur where $L'(x) = 0$

$L(x) = 9x + \frac{144}{x} = 9x + 144x^{-1}$

$L'(x) = 9 - 144x^{-2}$

$= 9 - \frac{144}{x^2}$

Let $L'(x) = 0$ to give

$9 - \frac{144}{x^2} = 0$

$9x^2 - 144 = 0$

$x^2 - 16 = 0$

$(x - 4)(x + 4) = 0$

$x = -4, x = 4$

Since x is a length it cannot be negative so $x = 4$.

We have to show that $x = 4$ gives a minimum value of $L(x)$.

x	1	4	10
$L'(x)$	$-$	0	$+$
Shape			

So, $x = 4$ gives a minimum value of $L(x)$.

2018 Paper 2 Question 9, (6)

$$P(x) = 2x + \frac{128}{x}$$

$$= 2x + 128x^{-1}$$

$$P'(x) = 2 - 128x^{-2} = 2 - \frac{128}{x^2}$$

For minimum values, $P'(x) = 0$.

$$2 - \frac{128}{x^2} = 0$$

$$2x^2 - 128 = 0$$

$$x^2 = 64$$

$$x = -8, x = 8$$

Discard $x = -8$ since x cannot be negative.

x	1	8	10
$P'(x)$	$-$	0	$+$
Shape	\	—	/

Substitute $x = 8$ to give

$$P(x) = (2\ x\ 8) + \frac{128}{8}$$

$$= 16 + 16$$

$$= 32$$

2019 Paper 2 Question 11, (3) (6)

a) Sides $= 4 \cdot 3x \cdot h$

$$= 12xh$$

Top $= (3x)^2 - x^2$

$$= 9x^2 - x^2$$

$$= 8x^2$$

Base $= 8x^2$

Inside of Tunnel $= 4xh$

Total $= 16x^2 + 12xh + 4xh$

$= 16x^2 + 16xh$

Volume of cuboid including central tunnel $= l \cdot b \cdot h$

$= (3x) \cdot (3x) \cdot h$

$= 9x^2 h$

Volume of tunnel $= l \cdot b \cdot h$

$= x^2 h$

Volume of cuboid without central tunnel $= 9x^2 h - x^2 h$

$= 8x^2 h$

Since volume $= 2000$ we have

$2000 = 8x^2 h$

$h = \frac{2000}{8x^2}$

Substituting into (1) gives Total Surface Area $= 16x^2 + 16x(\frac{2000}{8x^2})$

$= 16x^2 + \frac{4000}{x}$

So, $A(x) = 16x^2 + \frac{4000}{x}$.

b) $A(x) = 16x^2 + \frac{4000}{x}$

$= 16x^2 + 4000x^{-1}$

For minimum areas $A'(x) = 0$ giving

$32x - 4000x^{-2} = 0$

$32x - \frac{4000}{x^2} = 0$

$32x^3 - 4000 = 0$

$$x^3 = \frac{4000}{32}$$

$$x = \sqrt[3]{\frac{4000}{32}}$$

$$x = 5$$

x	1	5	10
$A'(x)$	$-$	0	$+$
Shape	$\searrow$	$\rightarrow$	$\nearrow$

The nature table confirms that $x = 5$ gives a minimum value of $A(x)$.

$$A(5) = 16 \cdot 5^2 + \frac{4000}{5}$$

$$= 400 + 800$$

$$= 1200\ cm^2$$

2022 Paper 2 Question 8, (3) (6)

a) Area of Pond $=$ *length x breadth*

$$= (x - 2 \cdot 1.5)(y - 2 \cdot 1)$$

$$= (x - 3)(y - 2) \qquad (1)$$

Area of Path & Pond $= xy$

So, $xy = 150$

$$y = \frac{150}{x}$$

Substituting into (1) gives

Area of Pond $= (x - 3)(\frac{150}{x} - 2)$

$$= 150 - 2x - \frac{450}{x} + 6$$

$$= 150 - 2x - \frac{450}{x} + 6$$

$$= 156 - 2x - \frac{450}{x}$$

So, $A(x) = 156 - 2x - \frac{450}{x}$

b) $A(x) = 156 - 2x - \frac{450}{x}$

$$= 156 - 2x - 450x^{-1}$$

$A'(x) = -2 + 450x^{-2}$

$$= -2 + \frac{450}{x^2}$$

Maximum area occurs where $A'(x) = 0$, giving

$-2 + \frac{450}{x^2} = 0$

$\frac{450}{x^2} = 2$

$450 = 2x^2$

$x^2 = 225$

$x = 15$ (discard $x = -15$ since it is a length and cannot be negative)

x	10	15	20
$A'(x)$	$+$	0	$-$
Shape	/	—	\

This shows that $x = 15$ gives a maximum value of $A(x)$.

$A(15) = 156 - 2(15) - \frac{450}{15}$

$$= 96$$

So, the maximum area is 96 metres squared.

2023 Paper 2 Question 14, (1) (2) (4)

a) i) Area $= (2x \cdot 3x) + 2(h \cdot 2x) + 2(h \cdot 3x)$

$= 6x^2 + 4hx + 6hx$

$= 6x^2 + 10hx$

ii) Volume $= length\ x\ breadth\ x\ height$

$= 3x \cdot 2x \cdot h$

$= 6hx^2$

Area $= 7200$ and Area $= 6x^2 + 10hx$

Equating gives

$6x^2 + 10hx = 7200$

$10hx = 7200 - 6x^2$

$h = \frac{7200 - 6x^2}{10x}$

Substitute $h = \frac{7200 - 6x^2}{10x}$ into $6hx^2$ to give

Volume $= 6x^2(\frac{7200 - 6x^2}{10x})$

$= \frac{43200x^2 - 36x^4}{10x}$

$= 4320x - \frac{18}{5}x^3$

b) $V = 4320x - \frac{18}{5}x^3$

$v'(x) = 4320 - \frac{54}{5}x^2$

Maximum volume occurs where $v'(x) = 0$ giving

$4320 - \frac{54}{5}x^2 = 0$

$21600 - 54x^2 = 0$

$54x^2 = 21600$

$x^2 = 400$

$x = 20$ (x cannot be -20 since it is a length)

x	10	20	30
$V'(x)$	$+$	0	$-$
$Shape$	/	—	\

So, $x = 20$ does give a maximum volume of the box.

2025 Paper 2 Question 10, (3) (6)

a) $P = 4x + 3x + 2y + 5x$

$P = 12x + 2y \qquad (1)$

Area rectangle $= y \times 5x = 5xy$

Area triangle $= \frac{1}{2} \times 3x \times 4x = 6x$

Total area $= 5xy + 6x$

So $5xy + 6x = 150$

$5xy = 150 - 6x$

$y = \frac{150 - 6x}{5x}$

Substitute into (1) to give

$P = 12x + 2(\frac{150 - 6x}{5x})$

$P = 12x + \frac{300}{5x} - \frac{12x}{5x}$

$P = 12x + \frac{60}{x} - \frac{12}{5}$

$P = 9.6x + \frac{60}{x}$

b) $P(x) = 9.6x + \frac{60}{x}$

$P(x) = 9.6x + 60x^{-1}$

$P'(x) = 9.6 - 60\,x^{-2}$

$P'(x) = 9.6 - \frac{60}{x^2}$

For minimum values $P'(x) = 0$ giving

$9.6 - \frac{60}{x^2} = 0$

$9.6x^2 = 60$

$x^2 = 6.25$

$x = 2.5$

Justify that this gives a minimum value using the second derivative

$P''(x) = 120\, x^{-3} = \frac{120}{x^3}$

$P''(2.5) = \frac{120}{2.5^3} = 7.68$

Since $P''(2.5) > 0, x = 2.5$ gives a minimum value of P.

Substitute $x = 2.5$ into $P(x) = 9.6x + \frac{60}{x}$ to give

$P(2.5) = 9.6(2.5) + \frac{60}{2.5}$

$\quad\quad = 48$

So, the minimum value of P is $48cm$.