Series

Solutions

Partial Sums of an Infinite Series

Q1) a) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ..$

$S_1 = \frac{1}{2},\ S_2 = \frac{3}{4},\ S_3 = \frac{7}{8}, ..$

The sequence of partial sums, S_n, is

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, ..$

$S_n = \frac{2^n - 1}{2^n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$

Since the sequence of partial sums converges, the series converges.

b) $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + . .$

$S_1 = 1,\ S_2 = 2,\ S_3 = 3, ..$

$S_n = n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$

Since the sequence of partial sums diverges, the series diverges.

c) $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n}) = \sum_{n=1}^{\infty} \ln(\frac{n+1}{n}) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$

$= [\ln 2 - \ln 1] + [\ln 3 - \ln 2] + [\ln 4 - \ln 3] + ..$

$S_1 = \ln 2\,,\ S_2 = \ln 3\,,\ S_3 = \ln 4\,, ..$

$S_n = \ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$

Since the sequence of partial sums diverges, the series diverges.

d) $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ..$

$= 1 - \frac{1}{n+1}$

The series of partial sums is given by

$S_n = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$

Since the sequence of partial sums converges, the series converges.

The Sum of a Geometric Series

Q1) a) $\sum_{n=0}^{\infty}(\frac{1}{2})^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ..$

This is a geometric series with $a = 1, r = \frac{1}{2}$

$\sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$

b) $\sum_{n=0}^{\infty}(-\frac{1}{2})^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ..$

This is a geometric series with $a = 1, r = -\frac{1}{2}$

$\sum_{n=0}^{\infty}(-\frac{1}{2})^n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$

c) $1 + 0.1 + 0.01 + ..$

This is a geometric series with $a = 1, r = \frac{1}{10}$

$1 + 0.1 + 0.01 + .. = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$

d) $\sum_{n=0}^{\infty} 3(\frac{3}{2})^n = 3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + ..$

This is a geometric series with $a = 3, r = \frac{3}{2}$

Since $r > 1$ the series diverges so has no limit.

The Sum of a Telescoping Series

Q1) a) $\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1}) = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+..$

$$= 1-\lim_{n\to\infty}\frac{1}{n+1} = 1-0 = 1$$

b) $\sum_{n=1}^{\infty}\frac{2}{4n^2-1} = \sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{2n+1})$ using partial fractions

$$= 1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+..$$

$$= 1-\lim_{n\to\infty}\frac{1}{2n+1} = 1-0 = 1$$

c) $\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \sum_{n=2}^{\infty}\frac{1}{2(n-1)}-\frac{1}{2(n+1)}$ using partial fractions

$$= \frac{1}{2}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\frac{1}{6}-\frac{1}{10}+\frac{1}{8}-\frac{1}{12}+\frac{1}{10}-\frac{1}{14}$$

$$= \frac{1}{2}+\frac{1}{4}-\lim_{n\to\infty}\frac{1}{2(n+1)} = \frac{1}{2}+\frac{1}{4}+0 = \frac{3}{4}$$

d) $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty}(\frac{1}{n+1}-\frac{1}{n+2})$ using partial fractions

$$= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+..$$

$$= \frac{1}{2}-\lim_{n\to\infty}\frac{1}{n+2} = \frac{1}{2}-0 = \frac{1}{2}$$

The Limit & Sum of an Infinite Series

Q1) a) $\sum_{n=1}^{\infty}\frac{n+10}{10n+1} = \sum_{n=1}^{\infty}\frac{1+\frac{10}{n}}{10+\frac{1}{n}}$

$$\lim_{n\to\infty}\frac{1+\frac{10}{n}}{10+\frac{1}{n}} = \frac{1}{10}$$

Since the limit of the n^{th} term is not 0, the series diverges.

b) $\sum_{n=1}^{\infty}\frac{n+1}{2n-1} = \sum_{n=1}^{\infty}\frac{1+\frac{1}{n}}{2-\frac{1}{n}}$

$$\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}$$

Since the limit of the n^{th} term is not 0, the series diverges.

c) $\sum_{n=0}^{\infty} \frac{4}{2^n}$

$$\lim_{n \to \infty} \frac{4}{2^n} = 0$$

Since the limit of the n^{th} term is 0 the n^{th} term test is inconclusive.

Use another test to check convergence / divergence.

d) $\sum_{n=0}^{\infty} \frac{1}{4^n}$

$$\lim_{n \to \infty} \frac{1}{4^n} = 0$$

Since the limit of the n^{th} term is 0 the n^{th} term test is inconclusive.

Use another test to check convergence / divergence.

Series Convergence Tests

Q1) a) $\sum_{n=1}^{\infty} \frac{1}{n+1}$

$$\int_1^{\infty} \frac{1}{x+1}\, dx = \lim_{a \to \infty} \int_1^{a} \frac{1}{x+1}\, dx$$

$$= \lim_{a \to \infty} [\ln|x+1|]_1^a$$

$$= \lim_{a \to \infty} (\ln a - \ln 2)$$

$$= \lim_{a \to \infty} (\ln a) - \ln 2$$

$$= \infty$$

Since the integral diverges the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ also diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$\int_1^{\infty} \frac{1}{x^3}\, dx = \lim_{a \to \infty} \int_1^{a} \frac{1}{x^3}\, dx$$

$$= \lim_{a \to \infty} \int_1^{a} x^{-3}\, dx$$

$$= \lim_{a \to \infty} \left[-\frac{1}{2}x^{-2}\right]_1^a$$

$$= \lim_{a \to \infty} \left[-\frac{1}{2x^2}\right]_1^a$$

$$= \lim_{a \to \infty} \left(\frac{1}{2} - \frac{1}{2a^2}\right)$$

$$= \frac{1}{2} - \lim_{a \to \infty} \frac{1}{2a^2} = \frac{1}{2}$$

Since the integral converges the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ also converges.

c) $\sum_{n=1}^{\infty} e^{-n}$

$$\int_1^{\infty} x^{-n}\ dn = \lim_{a\to\infty} \int_1^{a} e^{-x}\, dx$$
$$= \lim_{a\to\infty} [-e^{-x}]_1^a$$
$$= \lim_{a\to\infty} (-e^{-a} + e^1)$$
$$= e - \lim_{a\to\infty} \frac{1}{e^a}$$
$$= e - 0 = e$$

Since the integral converges the series $\sum_{n=1}^{\infty} e^{-n}$ also converges.

d) $\sum_{n=1}^{\infty} \frac{1}{4n+1}$

$$\int_1^{\infty} \frac{1}{4x+1}\, dx = \lim_{a\to\infty} \int_1^{a} \frac{1}{4x+1}\, dx$$
$$= \lim_{a\to\infty} \left[\frac{1}{4}\ln|4x+1|\right]_1^a$$
$$= \lim_{a\to\infty} (\frac{1}{4}\ln|4a+1| - \frac{1}{4}\ln 5)$$
$$= \frac{1}{4} \lim_{a\to\infty} \ln|4a+1| - \frac{1}{4}\ln 5$$
$$= \infty$$

Since the integral diverges the series $\sum_{n=1}^{\infty} \frac{1}{4n+1}$ also diverges.

Q2) a) $\sum_{n=1}^{\infty} \frac{1}{n}$

This is a p-series with $p = 1$. Since $0 < p \leq 1$ the series diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

This is a p-series with $p = 2$. Since $p > 1$ the series converges.

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{5}}}$

This is a p-series with $p = \frac{1}{5}$. Since $0 < p \leq 1$ the series diverges.

d) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + .. = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

This is a p-series with $p = \frac{1}{2}$. Since $0 < p \leq 1$ the series diverges.

Q3) a) $\sum_{n=0}^{\infty} 2^n$

$\lim_{n \to \infty} 2^n = \infty$

Since the limit of the n^{th} term is not 0, the series diverges.

b) $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$

$\lim_{n \to \infty} \frac{n!}{2n! + 1} = \lim_{n \to \infty} \frac{1}{2 + \frac{1}{n!}} = \frac{1}{2}$

Since the limit of the n^{th} term is not 0, the series diverges.

c) $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n \to \infty} \frac{1}{n} = 0$

Since the limit of the n^{th} term is 0, the n^{th} term test does not apply and you need another test to check convergence / divergence. This particular series, the harmonic series, is a p-series with $p = 1$, so it diverges using the p-series test.

d) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1$

Since the limit of the n^{th} term is not 0, the series diverges.

Q4) a) $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$

Compare with the series $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ..$

This is a geometric series with $r = \frac{1}{3}$, so it converges

$\frac{1}{2 + 3^n} < \frac{1}{3^n}$

So, by the direct comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$ converges.

b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Compare with the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ..$

This is a p-series with $p = 2$. Since $p > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\frac{1}{n^2+1} < \frac{1}{n^2}$

So, by the direct comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

c) $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} +$

$n! = n(n-1)(n-2)\ldots..3 \cdot 2 \cdot 1 \geq 2 \cdot 2 \cdot 2 \ldots..2 \cdot 2 \cdot 1 = 2^{n-1}$

So, $n! \geq 2^{n-1}$ for $n \geq 1$

So, $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$

Compare with the series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ..$

This is a geometric series with $r = \frac{1}{2}$, so it converges

$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$

So, by the direct comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

d) $\sum_{n=0}^{\infty} e^{-n^2} = \sum_{n=0}^{\infty} \frac{1}{e^{n^2}} = 1 + \frac{1}{e} + \frac{1}{e^4} + \frac{1}{e^9} + ..$

Compare with the series $\sum_{n=0}^{\infty} \frac{1}{e^n} = 1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3}$ ..

This is a geometric series with $r = \frac{1}{e}$, so it converges

$\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$

So, by the direct comparison test, the series $\sum_{n=0}^{\infty} e^{-n^2}$ converges.

Q5) a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Compare with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{n^2+1} / \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

Since the limit is positive and finite, both series diverge.

b) $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Compare with the convergent p-series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \Big/ \frac{\sqrt{n}}{n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

Since the limit is positive and finite, both series converge.

c) $\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$

Compare with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n+3}{n(n+2)} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n+3}{n+2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1+\frac{2}{n}} = 1$$

Since the limit is positive and finite, both series diverge.

d) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Compare with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \sin\frac{1}{n} \ / \ \frac{1}{n}$$

But as $n \to \infty$, $\sin\frac{1}{n} \to \sin 0 = 0$

And as $n \to \infty$, $\frac{1}{n} \to 0$, giving the indeterminate form $\frac{0}{0}$

Using L'Hospital's Rule, $\frac{\frac{d}{dx}(\sin(\frac{1}{x}))}{\frac{d}{dx}(\frac{1}{x})} = \frac{(\cos\frac{1}{x})\cdot(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \cos\frac{1}{x}$

Then, $\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1$

Since the limit is positive and finite, both series diverge.

Q6) a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$a_n = \frac{2^n}{n!}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Since $\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges.

b) $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$a_n = \frac{n!}{3^n}, a_{n+1} = \frac{(n+1)!}{3^{n+1}}$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left[\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right]$$

$$= \lim_{n \to \infty} \frac{n+1}{3} = \infty$$

Since $\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the series diverges.

c) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$a_n = \frac{(-1)^n 2^n}{n!}, a_{n+1} = \frac{(-1)^{n+1} 2^{n+1}}{(n+1)!}$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left[\frac{(-1)^{n+1} 2^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n 2^n}\right]$$

$$= \lim_{n \to \infty} \left[-\frac{2}{n+1}\right] = 0$$

Since $\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges.

d) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$a_n = \frac{\sqrt{n}}{n+1}, a_{n+1} = \frac{\sqrt{n+1}}{n+2}$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left[\frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}}\right]$$

$$= \lim_{n \to \infty} \left[\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n+1}{n+2}\right]$$

$$= \lim_{n \to \infty} \left[\sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}\right]$$

$$= \sqrt{1}(1) = 1$$

Since $\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive. Use another test to determine whether the series converges / diverges.

Q7) a) $\sum_{n=1}^{\infty}(\frac{n}{2n+1})^n$

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n+1}$$

$$= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

Since the limit is < 1 the series converges.

b) $\sum_{n=1}^{\infty}\frac{e^{2n}}{n^n}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty}\frac{e^2}{n} = 0$$

Since the limit is < 1 the series converges.

c) $\sum_{n=1}^{\infty}(\frac{2n}{n+1})^n$

$$\lim_{n\to\infty}\sqrt[n]{(\frac{2n}{n+1})^n} = \lim_{n\to\infty}\frac{2n}{n+1}$$

$$= \lim_{n\to\infty}\frac{2}{1+\frac{1}{n}} = 2$$

Since the limit is > 1 the series diverges.

d) $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(-1)^n}{(\ln n)^n}} = \lim_{n\to\infty}\frac{-1}{\ln n} = 0$$

Since the limit is < 1 the series converges.

Q8) a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

$a_n = \frac{1}{n}, a_{n+1} = \frac{1}{n+1}$

$a_{n+1} \leq a_n$

$\lim_{n\to\infty}\frac{1}{n} = 0$

So, by the alternating series test, the series converges.

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

$a_n = \frac{1}{2n-1}, a_{n+1} = \frac{1}{2n+1}$

$a_{n+1} \leq a_n$

$\lim_{n \to \infty} \frac{1}{n} = 0$

So, by the alternating series test, the series converges.

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

$a_n = \frac{1}{\sqrt{n}}, a_{n+1} = \frac{1}{\sqrt{n+1}}$

$a_{n+1} \leq a_n$

$\lim_{n \to \infty} \frac{1}{n} = 0$

So, by the alternating series test, the series converges.

d) $\sum_{n=1}^{\infty} \frac{(-1)^{n} n^2}{n^2+1}$

$a_n = \frac{n^2}{n^2+1}, a_{n+1} = \frac{(n+1)^2}{(n+1)^2+1}$

$\lim_{n \to \infty} \frac{n^2}{n^2+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} = 1$

Since $\lim_{n \to \infty} a_n \neq 0$, the alternating series test is inconclusive.

Consider using another test to determine convergence / divergence.