Integration Techniques

Solutions

The Power Rule

Q1) a) $\int x^2\,dx = \frac{1}{3}x^3 + c$

b) $\int 2x^3\,dx = \frac{2}{4}x^4 + c = \frac{1}{2}x^4 + c$

c) $\int (x^2 + 3)\,dx = \frac{1}{3}x^3 + 3x + c$

d) $\int 2x^{-3}\,dx = \frac{2}{-2}x^{-2} + c = -x^{-2} + c$

Q2) a) $\int -x^{-3}\,dx = \frac{1}{2}x^{-2} + c$

b) $\int -\frac{1}{2}x^3 + 2x\,dx = -\frac{1}{8}x^4 + x^2 + c$

c) $\int \frac{x^{-4}}{-2}\,dx = \int -\frac{1}{2}x^{-4}\ \,dx$

$= \frac{1}{6}x^{-3} + c$

d) $2x^{-3} + \frac{1}{4}x^3\,dx = \frac{2}{-2}x^{-2} + \frac{1}{16}x^4 + c$

Q3) a) $\int \frac{5}{x^4}\,dx = \int 5x^{-4}\,dx = \frac{5}{-3}x^{-3} + c$

$= -\frac{5}{3}x^{-3} + c$

b) $\int \frac{3}{2x^2}\,dx = \int \frac{3}{2}x^{-2}\,dx$

$= -\frac{3}{2}x^{-1} + c$

c) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + c$

d) $\int 2\sqrt[4]{x}\,dx = \int 2x^{\frac{1}{4}}\,dx = \frac{2}{\frac{5}{4}}x^{\frac{5}{4}} + c$

$= \frac{8}{5}x^{\frac{5}{4}} + c$

Definite Integrals

Q1) a) $\int_1^3 2x + 4\, dx = [x^2 + 4x]_1^3 = (3^2 + 4 \cdot 3) - (1^2 + 4 \cdot 1)$

$$= 21 - 5 = 16$$

b) $\int_0^4 6 - 3x\, dx = \left[6x - \frac{3}{2}x^2\right]_0^4 = \left(6 \cdot 4 - \frac{3}{2} \cdot 4^2\right) - (0)$

$$= 0 - 0 = 0$$

c) $\int_{-1}^2 3x^2 + 5x\, dx = \left[x^3 + \frac{5}{2}x\right]_{-1}^2 = \left(2^3 + \frac{5}{2} \cdot 2\right) - \left((-1)^3 + \frac{5}{2} \cdot -1\right)$

$$= 13 + 3.5 = 16.5$$

d) $\int_{-2}^2 2x^3\, dx = \left[\frac{1}{2}x^4\right]_{-2}^2 = \left(\frac{1}{2} \cdot 2^4\right) - (\frac{1}{2} \cdot (-2)^4)$

$$= 8 - 8 = 0$$

e) $\int_0^3 2x + 1\, dx = [x^2 + x]_0^3 = (3^2 + 3) - (0)$

$$= 12 - 0 = 12$$

f) $\int_{-2}^1 5 - x^3\, dx = \left[5x - \frac{1}{4}x^4\right]_{-2}^1 = \left(5 \cdot 1 - \frac{1}{4} \cdot 1^4\right) - (5 \cdot (-2) - \frac{1}{4}(-2)^4)$

$$= 4.75 + 14 = 18.75$$

Integration by U-Substitution

Q1) a) Let $\mathrm{I} = \int x\sqrt{1 + x}\ dx$

$u = 1 + x$, giving $\frac{du}{dx} = 1$

Substitute to give

$\mathrm{I} = \int (u - 1)u^{\frac{1}{2}}\, du$

$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$

$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c$

$= \frac{2(1+x)^{\frac{5}{2}}}{5} - \frac{2(1+x)^{\frac{3}{2}}}{3} + c$

b) Let $I = \int \frac{2}{\sqrt{x}(x-4)}\, dx$

$u = \sqrt{x}$, giving $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ or $2\, du = \frac{dx}{\sqrt{x}}$

Substitute to give

$I = \int \frac{4}{(u^2-4)} du$

$= \int \frac{4}{(u-2)(u+2)} du$

$= \int (\frac{1}{u-2} - \frac{1}{u+2}) du$

$= ln(u-2) - ln(u+2) + c$

$= ln(\frac{u-2}{u+2}) + c$

$= ln(\frac{\sqrt{x}-2}{\sqrt{x}+2}) + c$

c) Let $I = \int sin^3 x\, dx$

$u = cosx$, giving $\frac{du}{dx} = -sinx$ or $-du = \sin x\, dx$

$sin^3 x = sin\, x(1 - cos^2 x)$ since $sin^2 x + cos^2 x = 1$

$I = \int sin\, x(1 - cos^2 x) dx$

Substitute to give

$I = \int (u^2 - 1) du$

$= \frac{u^3}{3} - u + c$

$= \frac{cos^3 x}{3} - cos\, x + c$

d) Let $I = \int sec^4 x\, dx$

$u = tan\, x$, giving $\frac{du}{dx} = sec^2 x$ or $du = sec^2 x\, dx$

$sec^2 x = 1 + tan^2 x$ from trigonometric identities

$I = \int (1 + tan^2 x) sec^2 x\, dx$

Substitute to give

$I = \int (1 + u^2) du$

$= u + \frac{u^3}{3} + c$

$= tan\, x + \frac{tan^3 x}{3} + c$

The Chain Rule

Q1) a) $\int (x+4)^2\,dx = \frac{1}{3}(x+4)^3 + c$

b) $\int (2x+5)^3\,dx = \frac{1}{8}(2x+5)^4 + c$

c) $\int (2x^2-5)^2\,dx = \frac{1}{12x}(2x^2-5)^3 + c$

d) $\int (x-4)^7\,dx = \frac{1}{8}(x-4)^8 + c$

e) $\int \left(\frac{2x}{3}+5\right)^2 dx = \frac{1}{2}(\frac{2x}{3}+5)^3 + c$

f) $\int (3x^2-5)^{-3}\,dx = -\frac{1}{12x}(3x^2-5)^{-2} + c$

g) $\int (3-4x)^3\,dx = -\frac{1}{16}(3-4x)^4 + c$

h) $\int (5-3x^3)^{-2}\,dx = \frac{1}{9x^2}(5-3x^2)^{-1} + c$

Q2) a) $\int \frac{1}{(2x-3)^{-2}}\,dx = \int (2x\ -\ 3)^2\,dx$

$= \frac{1}{6}(2x-3)^3 + c$

b) $\int \frac{2}{(5-3x)^{-3}}\,dx = \int 2(5-3x)^3\,dx$

$= -\frac{1}{6}(5-3x)^4 + c$

c) $\int \frac{1}{(3x-2)^2}\,dx = \int (3x-2)^{-2}\,dx$

$= -\frac{1}{3}(3x-2)^{-1} + c$

d) $\int \sqrt{(5x-4)}\,dx = \int (5x-4)^{\frac{1}{2}}\,dx$

$= \frac{2}{15}(5x-4)^{\frac{3}{2}} + c$

Integration by Parts

Q1) a) $\int x\sin x\ \ dx$

Let $u = x,\ \frac{du}{dx} = 1$, giving $du = dx$

Let $\frac{dv}{dx} = \sin x\,,\ v = -\cos x$

$\int x\sin x\,dx = -x\cos x — \cos x(1)\,dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

b) $\int x\, e^x\, dx$

Let $u = x,\ \frac{du}{dx} = 1$, giving $du = dx$

Let $\frac{dv}{dx} = e^x,\ v = e^x$

$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + c$

c) $\int x \sec x \tan x\, dx$

Let $u = x\ , \frac{du}{dx} = 1$, giving $du = dx$

Let $\frac{dv}{dx} = \sec x \tan x\ ,\ v = \sec x$

$\int x \sec x \tan x\, dx = x \sec x - \int \sec x\, dx$

$= x \sec x - \ln|\sec x + \tan x| + c$

d) $\int \frac{\ln x}{x^3} dx$

Let $u = \ln x\ ,\ \frac{du}{dx} = \frac{1}{x}$, giving $du = \frac{1}{x}\ dx$

Let $\frac{dv}{dx} = x^{-3},\ v = \frac{x^{-2}}{-2}$

$\int \frac{\text{Ln}\, x}{x^3} dx = -\frac{1}{2x^2} \ln x — \frac{1}{2x^2}\left(\frac{1}{x}\right) dx$

$= -\frac{\ln x}{2x^2} + \int \frac{1}{2} x^{-3} dx$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$

e) $\int (x^2 + 1) \ln x\, dx$

Let $u = \ln x\ , \frac{du}{dx} = \frac{1}{x}$, giving $du = \frac{1}{x}\ dx$

Let $\frac{dv}{dx} = x^2 + 1\ , v = \frac{x^3}{3} + x$

$\int (x^2 + 1) \ln x\, dx = \ln x \left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^3}{3} + x\ \right)\left(\frac{1}{x}\right) dx$

$= \left(\frac{x^3}{3} + x\ \right) \ln x - \int \left(\frac{x^2}{3} + 1\right) dx$

$= \left(\frac{x^3}{3} + x\right) \ln x - \frac{x^3}{9} - x + c$

f) Let $\mathrm{I} = \int 12x^2(3 + 2x)^5 dx$

Let $u = 12x^2,\ \frac{du}{dx} = 24x$, giving $du = 24x\ dx$

Let $\frac{dv}{dx} = (3+2x)^5, v = \frac{(3+2x)^6}{12}$

$$\mathrm{I} = \int 12x^2(3+2x)^5 dx = 12x^2\left(\frac{(3+2x)^6}{12}\right) - \int 24x\frac{(3+2x)^6}{12} dx$$

$$= x^2(3+2x)^6 - \int 2x(3+2x)^6\, dx$$

Using integration by parts again

Let $\mathrm{J} = \int 2x(3+2x)^6\, dx$

Let $u = 2x, \frac{du}{dx} = 2$, giving $du = 2dx$

Let $\frac{dv}{dx} = (3+2x)^6, v = \frac{(3+2x)^7}{14}$

$$\mathrm{J} = \int 2x(3+2x)^6\, dx = x\frac{(3+2x)^7}{7} - \int \frac{(3+2x)^7}{7} dx$$

$$= x\frac{(3+2x)^7}{7} - \frac{(3+2x)^8}{112} + c$$

$$\mathrm{I} = x^2(3+2x)^6 - x\frac{(3+2x)^7}{7} - \frac{(3+2x)^8}{112} + c$$

Q2) a) $\int_0^{\ln 2} xe^{2x}\, dx$

Let $u = x,\ \frac{du}{dx} = 1$, giving $du = dx$

Let $\frac{dv}{dx} = e^{2x}, v = \frac{1}{2}e^{2x}$

$$\int_0^{\ln 2} xe^{2x}\, dx = \left[\frac{1}{2}e^{2x}(x)\right]_0^{\ln 2} - \int \frac{1}{2}e^{2x}\, dx$$

$$= \left(\frac{1}{2}e^{2\ln 2}\ \ln 2\right) - (0) - \left[\frac{1}{4}e^{2x}\right]_0^{\ln 2}$$

$$= \frac{4}{2}\ln 2 - \left(\left(\frac{1}{4}e^{2\ln 2}\right) - \left(\frac{1}{4}e^{0}\right)\right)$$

$$= 2\ln 2 - \frac{4}{4} + \frac{1}{4}$$

$$= 2\ln 2 - \frac{3}{4}$$

b) $\int_0^{\frac{\pi}{2}} x\sin x\, dx$

Let $u = x,\ \frac{du}{dx} = 1$, giving $du = dx$

Let $\frac{dv}{dx} = \sin x,\ v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x\sin x\, dx = [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(-\cos x)\, dx$$

$$= \left[-\frac{\pi}{2}\cos\frac{\pi}{2}\right] - (0) + \int_0^{\frac{\pi}{2}}(\cos x)\, dx$$

$$= 0 + [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1$$

Integration using Partial Fractions

Q1) a) $\frac{3x+5}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$

$3x + 5 \equiv A(x+2) + B(x+1)$

Let $x = -1 \Rightarrow 2 = A$

Let $x = -2 \Rightarrow B = 1$

$\int \frac{3x+5}{(x+1)(x+2)} dx = \int (\frac{2}{x+1} + \frac{1}{x+2})\, dx$

$= 2\ln|x+1| + \ln|x+1| + c$

$= \ln(|x+1|^2) + \ln|x+2| + c$

$= \ln|(x+1)^2(x+2)| + c$

b) $\frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$

$3x - 1 \equiv A(x-2) + B(2x+1)$

Let $x = 2 \Rightarrow 5 = 5B \Rightarrow B = 1$

Let $x = -\frac{1}{2} \Rightarrow -\frac{5}{2} = \frac{5}{2}A \Rightarrow A = 1$

$\int \frac{3x-1}{(2x+1)(x-2)} dx = \int (\frac{1}{2x+1} + \frac{1}{x-2})\, dx$

$= \frac{1}{2}\ln|2x+1| + \ln|x-2| + c$

$= \ln\left|(x-2)\sqrt{2x+1}\right| + c$

c) $\frac{2x-6}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$

$2x - 6 \equiv A(x-1) + B(x+3)$

Let $x = 1 \Rightarrow -4 = 4B \Rightarrow B = -1$

Let $x = -3 \Rightarrow -12 = -4A \Rightarrow A = 3$

$\int \frac{2x-6}{(x+3)(x-1)} dx = \int (\frac{3}{x+3} - \frac{1}{x-1}) dx$

$= 3\ln|x+3| - \ln|x-1| + c = \ln\left|\frac{(x+3)^3}{x-1}\right| + c$

d) $\frac{3}{(2+x)(1-x)}$

$\frac{3}{(2+x)(1-x)} \equiv \frac{A}{2+x} + \frac{B}{1-x}$

$3 \equiv A(1-x) + B(2+x)$

Let $x = 1 \Rightarrow 3 = 3B \Rightarrow B = 1$

Let $x = -2 \Rightarrow 3 = 3A \Rightarrow A = 1$

$\int \frac{3}{(2+x)(1-x)} dx = \int (\frac{1}{2+x} + \frac{1}{1-x}) dx$

$= \ln|2+x| - \ln|1-x| + c == \ln\left|\frac{2+x}{1-x}\right| + c$

Integrating the Trigonometric Functions

Q1) a) $\int 5\cos x\, dx = -5\sin x + c$

b) $\int 3\sin x\, dx = 3\cos x + c$

c) $\int -\frac{1}{3}\cos x\, dx = \frac{1}{3}\sin x + c$

d) $\int 2\cos(2x)\, dx = -\sin 2x + \mathrm{c}$

e) $\int 6\sin\left(\frac{3}{2}x\right) dx = 4\sin\left(\frac{3}{2}x\right) + c$

f) $\int -8\cos(4x)\, dx = 2\sin(4x) + c$

Improper Integrals

Q1) a) $\int_1^\infty \frac{1}{x^3} dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x^3} dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2}\right) = \frac{1}{2}$

$= \lim_{t\to\infty} \left(-\frac{1}{2t^2}\right) + \lim_{t\to\infty} (\frac{1}{2})$

$= 0 + \frac{1}{2} = \frac{1}{2}$

b) $\int_2^\infty x^{-\frac{3}{2}} dx$

$= \lim_{t\to\infty} \int_2^t x^{-\frac{3}{2}} dx$

$= \lim_{t\to\infty} \left[-2x^{-\frac{1}{2}}\right]_2^t$

$= \lim_{t\to\infty} (-2t^{-\frac{1}{2}} + \sqrt{2})$

$= \lim_{t\to\infty} (\frac{-2}{\sqrt{t}} + \sqrt{2})$

$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}}\right) + \lim_{t\to\infty} (\sqrt{2})$

$= 0 + \sqrt{2} = \sqrt{2}$

Q2) a) $\int_0^\infty e^x\, dx$

$= \lim_{t\to\infty} \int_0^t e^x dx$

$= \lim_{t\to\infty} [e^x]_0^t$

$= \lim_{t\to\infty} (e^t - 1)$

$= \lim_{t\to\infty}(e^t) + \lim_{t\to\infty}(-1)$

$e^t \to \infty$ as $t \to \infty$, so the integral diverges

b) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx = = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{2}}dx$

$= \lim_{t\to\infty}\left[2\sqrt{x}\right]_1^t$

$= \lim_{t\to\infty}(2\sqrt{t} - 2)$

$= \lim_{t\to\infty}\left(2\sqrt{t}\right) + \lim_{t\to\infty}(-2)$

$\sqrt{t} \to \infty$ as $t \to \infty$, so the integral diverges