The Chain Rule & Advanced Techniques Solutions

The Chain Rule

Second Derivatives

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Implicit Differentiation

The Chain Rule

Q1) a) $f(x) = (x+4)^3$

$f'(x) = 3(x+4)^2$

b) $f(x) = 3(x+1)^5$

$f'(x) = 15(x+1)^4$

c) $f(x) = (x-2)^{-3}$

$f'(x) = -3(x-2)^{-4}$

d) $f(x) = 4(x+2)^{-2}$

$f'(x) = -8(x+2)^{-3}$

Q2) a) $f(x) = (3x+2)^3$

$f'(x) = 3(3x+2)^2 \cdot 3$

$= 9(3x+2)^2$

b) $y = 2(2x-5)^6$

$\frac{dy}{dx} = 12(2x-5)^5 \cdot 2$

$= 24(2x-5)^5$

c) $f(x) = (4x-3)^{-2}$

$f'(x) = -2(4x-3)^{-3} \cdot 4$

$= -8(4x-3)^{-3}$

d) $g(x) = 2(1-3x)^2$

$g'(x) = 4(1-3x) \cdot -3$

$= -12(1-3x)$

Q3) a) $f(x) = (4x-4)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(4x-4)^{-\frac{1}{2}} \cdot 4$

$= 2(4x-4)^{-\frac{1}{2}}$

b) $f(x) = 3(2x+5)^{-\frac{1}{3}}$

$f'(x) = -(2x+5)^{-\frac{4}{3}} \cdot 2$

$= -2(2x+5)^{-\frac{4}{3}}$

c) $g(x) = \sqrt{(2-4x)^3}$

$= (2-4x)^{\frac{3}{2}}$

$g'(x) = \frac{3}{2}(2-4x)^{\frac{1}{2}} \cdot -4$

$= -6(2-4x)^{\frac{1}{2}}$

d) $h(x) = \sqrt[4]{(\frac{1}{2}x^2-5)^3}$

$= (\frac{1}{2}x^2-5)^{\frac{3}{4}}$

$h'(x) = \frac{3}{4}(\frac{1}{2}x^2-5)^{-\frac{1}{4}} \cdot x$

$= \frac{3}{4}x(\frac{1}{2}x^2-5)^{-\frac{1}{4}}$

Second Derivatives

Q1) a) $y = 12x^2 + 3x + 8$

$\frac{dy}{dx} = 24x + 3$

$\frac{d^2y}{dx^2} = 24$

b) $y = 15x + 6 + \frac{3}{x}$

$\frac{dy}{dx} = 15 - 3x^{-2}$

$\frac{d^2y}{dx^2} = 6x^{-3}$

c) $y = 9\sqrt{x} - \frac{3}{x^2} = 9x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{9}{2}x^{-\frac{1}{2}} + 6x^{-3}$

$\frac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{3}{2}} - 18x^{-4}$

d) $y = (5x+4)(3x-2)$

$= 15x^2 + 2x - 8$

$\frac{dy}{dx} = 30x + 2$

$\frac{d^2y}{dx^2} = 30$

Exponential & Logarithmic Functions

Q1) a) $y = 4e^{7x}$

$\frac{dy}{dx} = 28e^{7x}$

b) $y = e^{-3x-x^2}$

$\frac{dy}{dx} = (-3-2x)e^{-3x-x^2}$

c) $y = e^{1-x}$

$\frac{dy}{dx} = -e^{1-x}$

d) $y = e^{3x} - e^{-3x}$

$\frac{dy}{dx} = 3e^{3x} + 3e^{-3x}$

e) $y = e^{\sqrt{x}} = e^{x^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{\sqrt{x}}$

f) $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

Q2) a) $y = \ln(2x)$

$\frac{dy}{dx} = \frac{2}{2x}$

b) $y = \ln(x^3)$

$\frac{dy}{dx} = \frac{3x^2}{x^3}$

c) $y = \ln(5x - 2x^2)$

$\frac{dy}{dx} = \frac{5 - 4x}{5x - 2x^2}$

d) $y = \ln(\frac{1}{x}) = \ln(x^{-1})$

$\frac{dy}{dx} = \frac{-x^{-2}}{x^{-1}} = -\frac{1}{x}$

Q3) a) $y = \ln(e^{x^2})$

$\frac{dy}{dx} = \frac{2xe^{x^2}}{e^{x^2}} = 2x$

b) $y = \ln(3 + e^{2x})$

$\frac{dy}{dx} = \frac{2e^{2x}}{3 + e^{2x}}$

Implicit Differentiation

Q1) a) $x^2 + y^3 = 2$

$2x + 3y^2\frac{dy}{dx} = 0$

$3y^2\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{2x}{3y^2}$

b) $x^2 + 5y^2 = 14$

$2x + 10y\frac{dy}{dx} = 0$

$10y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{10y} = -\frac{x}{5y}$

c) $y^3 + 3x^2y - 4x = 0$

Using the product rule to give

$3y^2\frac{dy}{dx} + 3x^2\frac{dy}{dx} + y(6x) - 4 = 0$

$(3y^2 + 3x^2)\frac{dy}{dx} = 4 - 6xy$

$\frac{dy}{dx} = \frac{4 - 6xy}{3(y^2 + x^2)}$

d) $(x - y)^4 = x + y + 5$

Using the chain rule to give

$4(x - y)^3\left(1 - \frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$

$4(x - y)^3 - 4(x - y)^3\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$4(x - y)^3 - 1 = \frac{dy}{dx} + 4(x - y)^3\frac{dy}{dx}$

$4(x - y)^3 - 1 = (1 + 4(x - y)^3)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{4(x - y)^3 - 1}{1 + 4(x - y)^3}$

e) $e^x y = xe^y$

Using the product rule to give

$e^x \frac{dy}{dx} + ye^x = xe^y \frac{dy}{dx} + e^y$

$e^x \frac{dy}{dx} - xe^y \frac{dy}{dx} = e^y - ye^x$

$(e^x - xe^y)\frac{dy}{dx} = ey - ye^x$

$\frac{dy}{dx} = \frac{ey - ye^x}{e^x - xe^y}$

f) $\sqrt{xy} + x + y^2 = 0$

$x^{\frac{1}{2}}y^{\frac{1}{2}} + x + y^2 = 0$

Using the product rule to give

$\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}(x^{\frac{1}{2}}) + 1 + 2y\frac{dy}{dx} = 0$

$\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}(x^{\frac{1}{2}}) + 2y\frac{dy}{dx} = -1 - \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}$

$\frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} + 2y\frac{dy}{dx} = -1 - \frac{\sqrt{y}}{\sqrt{x}}$

$\left(\frac{\sqrt{x}}{2\sqrt{y}} + 2y\right)\frac{dy}{dx} = -1 - \frac{\sqrt{y}}{\sqrt{x}}$

$\frac{dy}{dx} = -\frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{\frac{\sqrt{x}}{2\sqrt{y}} + 2y}$