Applications

Solutions

The Equation of a Tangent to a Curve

Q1) a) $y = x^2 + 3x + 6$

Let $f(x) = x^2 + 3x + 6$

$f'(x) = 2x + 3$

$f'(1) = 5$

So, the gradient of the tangent at $x = 1$ is 5

Using $y - b = m(x - a)$ we have

$y - 10 = 5(x - 1)$

$y = 5x + 5$

b) $y = x + \frac{1}{x} = x + x^{-1}$

Let $f(x) = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

$f'(2) = \frac{3}{4}$

So, the gradient of the tangent at $x = 2$ is $\frac{3}{4}$

Using $y - b = m(x - a)$ we have

$y - \frac{5}{2} = \frac{3}{4}(x - 2)$

$y = \frac{3}{4}x + 1$

c) $y = 4\sqrt{x} = 4x^{\frac{1}{2}}$

Let $f(x) = 4x^{\frac{1}{2}}$

$f'(x) = 2x^{-\frac{1}{2}} = \frac{2}{\sqrt{x}}$

$f'(9) = \frac{2}{3}$

So, the gradient of the tangent at $x = 9$ is $\frac{2}{3}$

Using $y - b = m(x - a)$ we have

$y - 12 = \frac{2}{3}(x - 9)$

$y - 12 = \frac{2}{3}x - 6$

$y = \frac{2}{3}x + 6$

d) $y = \cos x$ at $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$

Let $f(x) = \cos x$

$f'(x) = -\sin x$

$f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

So, the gradient of the tangent at $x = \frac{\pi}{6}$ is $-\frac{1}{2}$

Using $y - b = m(x - a)$ we have

$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}(x - \frac{\pi}{6})$

$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}x + \frac{\pi}{12}$

$y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

Increasing & Decreasing Functions

Q1) a) $f(x) = x^3 - 2x^2 + 6x + 3$

$f'(x) = 3x^2 - 4x + 6$

$f'(1) = 5$

Since $f'(1) > 0$, the function is increasing at $x = 1$

b) $f(x) = 2x^3 + x^2 - 8x - 1$

$f'(x) = 6x^2 + 2x - 8$

$f'(-1) = -4$

Since $f'(-1) < 0$, the function is decreasing at $x = -1$

c) $f(x) = x^4 - 2x^3 - 2x^2 - 3$ when $x = 2$

$f'(x) = 4x^3 - 6x^2 - 4x$

$f'(2) = 0$

Since $f'(2) = 0$, the function is stationary at $x = 0$

Q2) $f(x) = x^2 - \frac{3}{x}$

$= x^2 - 3x^{-1}$

$f'(x) = 2x + 3x^{-2}$

$= 2x + \frac{3}{x^2}$

$f'(-1) = -2$

Since $f'(-1) < 0$, the function is decreasing at $x = -2$

Determining Local Extrema of a Function

Q1) a) $f(x) = 2x^4 - 12$

$f'(x) = 8x^3$

For SP's $f'(x) = 0$

$8x^3 = 0$

$x = 0$

When $x = 0, y = -12$

Stationary Point is at $(0, -12)$

b) $f(x) = 4x$

$f'(x) = 4$

Since the derivative is a constant there are no stationary points. This makes sense because the function, $4x$, is a straight line which has no stationary points.

c) $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

For SP's $\frac{dy}{dx} = 0$

$1 - x^{-2} = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1, y = 1 + \frac{1}{1} = 2$

When $x = -1, y = -1 + \frac{1}{-1} = -2$

Stationary points are $(1, 2)$ and $(-1, -2)$

d) $y = x - 3\sqrt{x} = x - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}} = 1 - \frac{3}{2\sqrt{x}}$

For SP's $\frac{dy}{dx} = 0$

$1 - \frac{3}{2\sqrt{x}} = 0$

$\frac{3}{2\sqrt{x}} = 1$

$\sqrt{x} = \frac{3}{2}, x = \frac{9}{4}$

When $x = \frac{9}{4}$

$y = \frac{9}{4} - 3\sqrt{\frac{9}{4}} = -\frac{9}{4}$

Stationary points is at $\left(\frac{9}{4}, -\frac{9}{4}\right)$

Q3) a) $y = \frac{1}{3}x^3 - 2x^2 + 3x - 1$

$\frac{dy}{dx} = x^2 - 4x + 3$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 1, x = 3$

When $x = 1, y = \frac{1}{3} \cdot 1^3 - 2 \cdot 1^2 + 3 \cdot 1 - 1 = \frac{1}{3}$

When $x = 3, y = (\frac{1}{3} \cdot 3^3) - (2 \cdot 3^2) + (3 \cdot 3) - 1 = -1$

Stationary points are at $(1, \frac{1}{3})$ and $(3, -1)$

x	0	1	2	3	4
$\frac{dy}{dx}$	+	0	−	0	+
Shape					

So, $(1, \frac{1}{3})$ is a maximum turning point and $(3, -1)$ is a minimum turning point.

b) $y = x^3 - 2x^2 - 4x + 1$

$\frac{dy}{dx} = 3x^2 - 4x - 4$

For stationary points $\frac{dy}{dx} = 0$

$3x^2 - 4x - 4 = 0$

$(3x + 2)(x - 2) = 0$

$x = -1.5, x = 2$

When $x = -1.5, y = 1.5^3 - (2 \cdot 1.5^2) - (4 \cdot 1.5) + 1 = -6.13$

When $x = 2, y = 2^3 - (2 \cdot 2^2) - (4 \cdot 2) + 1 = -7$

Stationary points are at $(-1.5, -6.13)$ and $(2, -7)$

x	-2	-1.5	1	2	3
$\frac{dy}{dx}$	+	0	−	0	+
Shape					

$(-1.5, -6.13)$ is a maximum turning point and $(2, -7)$ is a minimum turning point.

c) $y = 2 + 5x - x^2 - x^3$

$\frac{dy}{dx} = 5 - 2x - 3x^2$

For stationary points $\frac{dy}{dx} = 0$

$5 - 2x - 3x^2 = 0$

$3x^2 + 2x - 5 = 0$

$(3x + 5)(x - 1) = 0$

$x = -\frac{5}{3}, x = 1$

When $x = -\frac{5}{3}, y = 2 + \left(5 \cdot -\frac{5}{3}\right) - \left(-\frac{5}{3}\right)^2 - \left(-\frac{5}{3}\right)^3 = 13.7$

When $x = 1, y = 2 + (5 \cdot 1) - 1^2 - 1^3 = 5$

Stationary points are at $(-\frac{5}{3}, 13.7)$ and $(1, 5)$

x	-2	$-\frac{5}{3}$	0	1	2
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$
Shape					

$(-\frac{5}{3}, 13.7)$ is a minimum turning point and $(1, 5)$ is a maximum turning point.

d) $y = x^3(x - 2)$

$y = x^4 - 2x^3$

$\frac{dy}{dx} = 4x^3 - 6x^2$

For stationary points $\frac{dy}{dx} = 0$

$4x^3 - 6x^2 = 0$

$2x^2(2x - 3) = 0$

$x = 0, x = 1.5$

When $x = 0, y = 0^4 - 2 \cdot 0^3 = 0$

When $x = 1.5, y = 1.5^4 - 2 \cdot 1.5^3 = -1.69$

Stationary points are at $(0, 0)$ and $(1.5, -1.69)$

x	-1	0	1	1.5	2
$\frac{dy}{dx}$	$-$	0	$-$	0	$+$
Shape					

$(0, 0)$ is a falling point of inflection and $(1.5, -1.69)$ is a minimum turning point.

Q4) a) $y = 4x^2 + 6x$

$\frac{dy}{dx} = 8x + 6$

For stationary points $\frac{dy}{dx} = 0$

$8x + 6 = 0$

$8x = -6, x = -\frac{3}{4}$

When $x = -\frac{3}{4}$, $\frac{d^2y}{dx^2} = 8$

Since $\frac{d^2y}{dx^2} > 0$, the point is a minimum turning point.

b) $y = x(x^2 - 4x - 3) = x^3 - 4x^2 - 3x$

$\frac{dy}{dx} = 3x^2 - 8x - 3$

For stationary points $\frac{dy}{dx} = 0$

$3x^2 - 8x - 3 = 0$

$(3x + 1)(x - 3) = 0$

$x = -\frac{1}{3},\ x = 3$

$\frac{d^2y}{dx^2} = 6x - 8$

When $x = -\frac{1}{3}, \frac{d^2y}{dx^2} = -10$

Since $\frac{d^2y}{dx^2} < 0$, the point is a maximum turning point.

When $x = 3, \frac{d^2y}{dx^2} = 10$. Since $\frac{d^2y}{dx^2} > 0$, the point is a minimum turning point.

Sketching Curves

Q1) a) $y = 4x^2 + 6x$

$\frac{dy}{dx} = 8x + 6$

For stationary points $\frac{dy}{dx} = 0$

$8x + 6 = 0, x = -\frac{3}{4}$

When $x = -\frac{3}{4}, y = 4\left(-\frac{3}{4}\right)^2 + 6\left(-\frac{3}{4}\right) = -\frac{9}{4}$

So, we have a stationary point at $(-\frac{3}{4}, -\frac{9}{4})$

$\frac{d^2y}{dx^2} = 8$

Since $\frac{d^2y}{dx^2} > 0$, $(-\frac{3}{4}, -\frac{9}{4})$ is a minimum turning point

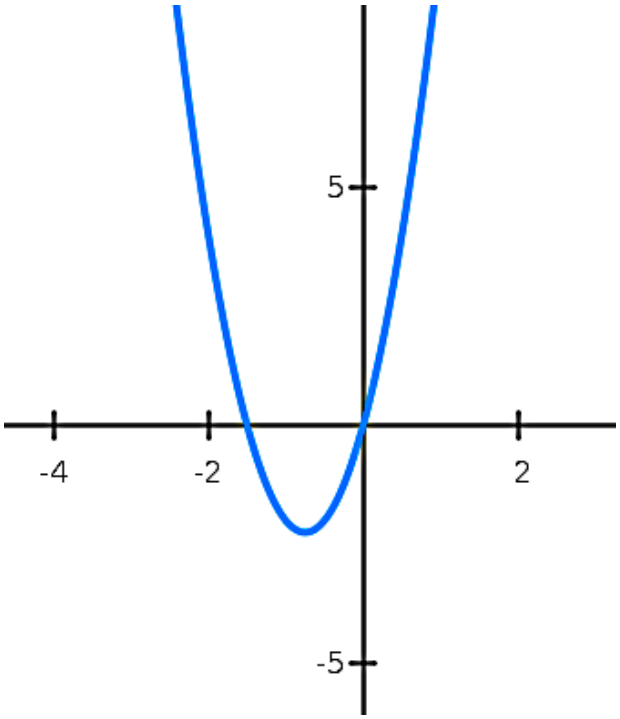
For the x intercepts let $y = 0$ to give

$4x^2 + 6x = 0$

$2x(x + 3) = 0$

$x = 0, x = -3$, giving $(0, 0)$ and $(-3, 0)$

For the y intercept let $x = 0$ to give $y = 0$, giving $(0, 0)$



b) $y = x^3 - 3x^2$

$\frac{dy}{dx} = 3x^2 - 6x$

For stationary points $\frac{dy}{dx} = 0$

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0, x = 2$

When $x = 0, y = 0.$

So, we have a stationary point at $(0, 0)$

When $x = 2, y = 2^3 - 3(2^2) = -4.$

So, we have a stationary point at $(2, -4)$

$\frac{d^2y}{dx^2} = 6x - 6$

When $x = 0, \frac{d^2y}{dx^2} = -6$

Since $\frac{d^2y}{dx^2} < 0$, $(0, 0)$ is a maximum turning point

When $x = 2, \frac{d^2y}{dx^2} = 6$

Since $\frac{d^2y}{dx^2} > 0$, $(2, -4)$ is a minimum turning point

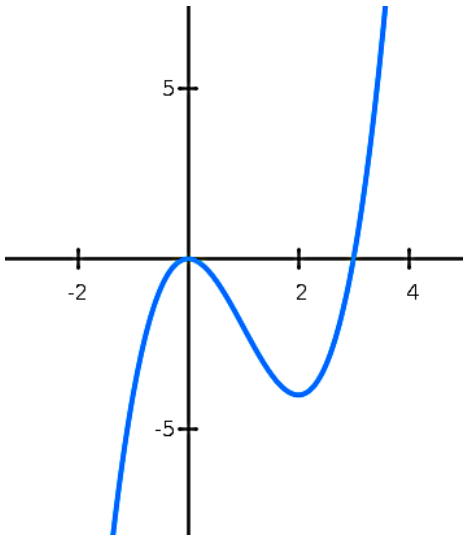
For the x intercepts let $y = 0$ to give

$x^3 - 3x^2 = 0$

$x^2(x - 3) = 0$

$x = 0, x = 3$, giving $(0, 0)$ and $(3, 0)$

For the y intercept let $x = 0$ to give $y = 0$



Optimization

Q1) a) Area of each side is given by xh

There are four sides, giving $4xh$

Area of the base is given by x^2

Total surface area is given by $x^2 + 4xh$ (1)

Volume is given by x^2h, but the volume is 500 cm^3, so we have

$x^2h = 500$

$h = \frac{500}{x^2}$

Substituting this in (1) gives

$x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$

So, $S(x) = x^2 + \frac{2000}{x}$

b) $S(x) = x^2 + \frac{2000}{x}$

$= x^2 + 2000x^{-1}$

$S'(x) = 2x - 2000x^{-2}$

$= 2x - \frac{2000}{x^2}$

Minimal areas occur where $S'(x) = 0$, giving

$2x - \frac{2000}{x^2} = 0$

$2x = \frac{2000}{x^2}$

$2x^3 = 2000$

$x^3 = 1000$

$x = 10$

x	1	10	20
$S'(x)$	$-$	0	$+$
Shape			

So, $x = 10$ produces a minimized surface area.

When $x = 10, h = \frac{500}{10^2} = 5$

Q2) a) $P(x) = 192x - x^3 + 400$

$P'(x) = 192 - 3x^2$

Maximum profits occur where $P'(x) = 0$ giving

$192 - 3x^2 = 0$

$3x^2 - 192 = 0$

$x^2 - 64 = 0$

$(x - 8)(x + 8) = 0$

$x = -8,\ x = 8$

Since x represents the number of jobs it cannot be negative, so $x = 8$

x	1	8	10
$S'(x)$	+	0	−
Shape			

So, $x = 8$ does produce the maximum profit as confirmed by the nature table.

b) When $x = 8, P = 192(8) - 8^3 + 400$

$= 1,424$

So, the maximum profit achieved in one day is £1,424

Rolle's Theorem & The Mean Value Theorem

Q1) a) $f(x) = x^2 - 3x$

f is continuous on $[0, 3]$ and $f(0) = 0 = f(3)$

$f'(x) = 2x - 3 = 0$

$2x = 3$

$x = \frac{3}{2}$

So, $c = \frac{3}{2}$

b) $f(x) = \frac{x^2 - 1}{x}$

f has a discontinuity at $x = 0$ so Rolle's Theorem cannot be used.

c) $f(x) = \sin 2x$

f is continuous on $[\frac{\pi}{6}, \frac{\pi}{3}]$ and $f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = f\left(\frac{\pi}{3}\right)$

$f'(x) = 2\cos 2x = 0$

$\cos 2x = 0$

$x = \frac{\pi}{4}$

So, $c = \frac{\pi}{4}$

Q2) a) $f(x) = x^2, f'(x) = 2x$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$2c = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$

$c = -\frac{1}{2}$

b) $f(x) = \sqrt{x - 2} = (x - 2)^{\frac{1}{2}}, f'(x) = \frac{1}{2}(x - 2)^{-\frac{1}{2}}$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\frac{1}{2}(c - 2)^{-\frac{1}{2}} = \frac{f(6) - f(2)}{6 - 2} = \frac{2 - 0}{4} = \frac{1}{2}$

$\frac{1}{2}(c - 2)^{-\frac{1}{2}} = \frac{1}{2}$

$\frac{1}{(c - 2)^{\frac{1}{2}}} = 1$

$(c - 2)^{\frac{1}{2}} = 1$

$c = 3$

c) $f(x) = \frac{x+1}{x} = 1 + x^{-1}, f'(x) = -x^{-2}$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$-c^{-2} = \frac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}} = \frac{\frac{3}{2} - 3}{\frac{3}{2}} = -1$

$-c^{-2} = -1$

$\frac{1}{c^2} = 1$

$c = 1$

d) $f(x) = \sin x, f'(x) = \cos x$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\cos c = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$

$c = \frac{\pi}{2}$